Partial Algebraic Quantum Groups and their Drinfeld Doubles

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Motivation I

A unitary tensor C^*-category is a *rigid* tensor C^*-category, with *irreducible* unit, closed under *subobjects*.

Examples

- Representation theory of compact quantum groups,
- Subfactor theory,
- Conformal Field Theory,
- ...

Motivation II

- Not all unitary tensor C^*-categories can be described as Rep($\mathbb{G}$), for some compact quantum group $\mathbb{G}$.

HOWEVER

- *Any* unitary tensor C^*-category is the representation category of a *partial* compact quantum group. See e.g. [DCT15] (continuing the work in [Hay93, Hay99])

Introduction

Why study partial compact quantum groups (PCQGs)?

- Are a sufficiently general structure to describe all C^*-tensor categories
- One may still use the techniques from CQGs in the study of general C^*-tensor categories.

Why generalise to partial algebraic quantum groups?

- Are the ‘right’ framework to study Drinfeld doubles of PCQGs
- Hope: Translating the language of C^*-tensor categories to that of (discrete) quantum groupoids might help in generalising these to (analytic) locally compact quantum groupoids

Examples of PAQGs

Examples

- Applying the Drinfeld double construction to PCQGs leads to non-trivial examples of PAQGs, e.g. the Drinfeld double of dynamical $SU_q(2)$,
- Direct sums of PAQGs are again PAQGs,
- Discrete groupoids give rise to PAQGs.

Overview

1. Preliminaries
2. Regular I-partial multiplier Hopf algebras
3. I-partial algebraic quantum groups (invariant functionals)
4. Duality
5. I-partial $*$-algebraic qauntum groups ($*$-structure)
6. Drinfeld double

I-partial algebras

Let I be a (possibly infinite) set.

Definition

An I-partial algebra A is an I^2-graded vector space, written as:

$$A = \bigoplus_{r,s \in I} {}_rA_s$$

along with a multiplication map

$$m : A \times A \to A, (a, b) \mapsto a \cdot b = ab,$$

such that, $\forall r, s, s', t \in I$:

$${}_rA_s \cdot {}_{s'}A_t \subseteq \delta_{s,s'} \, {}_rA_t.$$

The Multiplier algebra $M(A)$

Definition

$M(A)$ is a subalgebra of $\mathrm{End}(A) \oplus \mathrm{End}(A)^{op}$ defined by:

$$\left\{ m = (\lambda, \rho) \mid \forall a, b \in A : a \cdot \lambda(b) = \rho(a) \cdot b \right\}.$$

We write:

$$ma := \lambda(a), \qquad am := \rho(a).$$

We endow $M(A)$ with the *strict* topology:
$(m_\alpha)_\alpha \xrightarrow{strict} m$ in $M(A)$ if
$\forall a \in A : m_\alpha a = ma,\ am_\alpha = am$ eventually.

Examples

The multipliers $\mathbf{1}_r \in M(A)$ are defined by, $\forall a \in {}_{r'}A_{s'}$:

$$\mathbf{1}_r a = \delta_{r,r'} a, \qquad a \mathbf{1}_s = \delta_{s,s'} a$$

Technical conditions

We ask A to be non-degenerate and idempotent as a partial algebra.

Definition

A is called non-degenerate and idempotent as a partial algebra if for any $r, s \in I$
A_s is left non-degenerate and idempotent.
${}_rA$ is right non-degenerate and idempotent.

Remark

These conditions are stronger than non-degeneracy/idempotency of the total algebra A.

Morphisms

Let A and B again be I-partial algebras.

Definition

A *morphism* $A \xrightarrow{f} B$ is a homomorphism of algebras $f : A \to M(B)$, such that, for all $r, s \in I$: $B_s \cdot f(A_r) = B_s$ and $f\left({}_rA\right) \cdot {}_rB = {}_rB$

I^2-partial algebras

We usually consider I^2-partial algebras, writing I^2 $(= I \times I)$ as $\left\{ \binom{r}{s} \middle| r, s \in I \right\}$, e.g. the base units are $\mathbf{1}\binom{r}{s}$.

Remark

The following sums are strictly convergent:

$$\mathbf{1}_s = \sum_s \mathbf{1}\binom{r}{s}, \qquad \mathbf{1}^r = \sum_s \mathbf{1}\binom{r}{s}.$$

Tensor product of I^2-partial algebras

Let A, B be non-degenerate and idempotent I^2-partial algebras

Definition

The *tensor product* $A \otimes^I B$ is defined to be a subalgebra of $A \otimes B$:

$$A \otimes^I B := \bigoplus {}^r_t A^s_u \otimes {}^t_v B^u_w \subsetneq A \otimes B.$$

It is a (non-degenerate and idempotent) I^2-partial algebra for the grading:

$${}^r_v\left(A \otimes^I B\right)^s_w := \bigoplus_{t,u} {}^r_t A^s_u \otimes {}^t_v B^u_w.$$

The idempotent E

Definition

Set $E = \sum_s \mathbf{1}_s \otimes \mathbf{1}^s \in A \otimes B$, an idempotent in $A \otimes B$.
One could equivalently define $A \otimes^I B$ as a corner $E(A \otimes B)E$.

Remark

We have an isomorphism

$$E \cdot M(A \otimes B) \cdot E \xrightarrow{\cong} M\left(A \otimes^I B\right),$$

by restricting the action of multipliers.

The restricted multiplier algebra

Definition

We define the restricted multiplier algebra of $A \otimes^l B$ as the following subalgebra of $M(A \otimes^l B)$:

$$\widetilde{M}(A \otimes^l B) := \left\{ x \in M(A \otimes^l B) \middle| \begin{smallmatrix} (a\otimes 1)x,\ x(a\otimes 1) \\ (1\otimes b)x,\ x(1\otimes b) \end{smallmatrix} \in A \otimes B \right\}$$

Let now $\omega : B \to \mathbb{C}$ be a functional.

Remark

For any $x \in \widetilde{M}(A \otimes^l B)$, there is a unique element denoted $(\mathrm{id} \otimes \omega)(x) \in M(A)$ defined by:

$$(\mathrm{id} \otimes \omega)(x) \cdot a = (\mathrm{id} \otimes \omega)\left(x(a \otimes 1)\right),$$
$$a \cdot (\mathrm{id} \otimes \omega)(x) = (\mathrm{id} \otimes \omega)\left((a \otimes 1)x\right)$$

The Comultiplication Δ

Let A be an I^2-partial algebra.

Definition (comultiplication)

A morphism $A \xrightarrow{\Delta} A \otimes^I A$ is called a (regular) comultiplication (on A) if it satisfies:

- $\Delta(A) \subseteq \widetilde{M}(A \otimes^I A)$,
- $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$.

Remark

- Automatically, we get $\Delta(1) = E$.
- Regularity of Δ will always be implicitly assumed.

The Count ε

Let A still be an I^2-partial algebra. Let Δ be a comultiplication on A.

Definition (counit)

A functional $\varepsilon : A \to \mathbb{C}$ is called a counit for (A, Δ) if it satisfies:

1. For all $r, s, t, u \in I$, if $r \neq t$ or $s \neq u$: $\varepsilon\left({}_t^r A_u^s\right) = 0$,
2. For all $r, s \in I$: $\varepsilon\left({}_r A\right) \neq 0 \neq \varepsilon\left(A_s\right)$,
3. For all $s, t \in I$, for all $a \in A_t^s$, $b \in {}_t^s A$:
$$\varepsilon\left(ab\right) = \varepsilon\left(a\right)\varepsilon\left(b\right),$$
4. For all $a \in A$: $(\mathrm{id} \otimes \varepsilon)\Delta(a) = a = (\varepsilon \otimes \mathrm{id})\Delta(a)$.

The regular I-partial multiplier bialgebra (A, Δ, ϵ)

Definition (PMB)

A triple (A, Δ, ε), consisting of an I-partial algebra together with a regular comultiplication Δ on A and a counit ε for (A, Δ) is called a *regular I-partial multiplier bialgebra*.

- Allowing $\varepsilon({}_rA) = 0$ or $\varepsilon(A_r) = 0$, would imply allowing some of the base units $\mathbf{1}^r, \mathbf{1}_r$ to be zero.
- The compultiplication Δ of a regular I-partial multiplier bialgebra (A, Δ, ε) is always *full*.
- There can exist at most one counit.

First result

Proposition

Any regular I-partial multiplier bialgebra is a weak multiplier bialgebra, cf. [BG-TL-C15].

Partial multiplier Hopf algebras

Definition (PMHA)

A regular I-partial multiplier bialgebra is called a *regular I-partial multiplier Hopf algebra* if the following four maps are bijective:

- $\mathrm{can}_r : \bigoplus_r (A_r \otimes {}_rA) \to \bigoplus_r ({}_rA \otimes {}^rA), \qquad a \otimes b \mapsto \Delta(a)(1 \otimes b),$
- $\mathrm{can}_l : \bigoplus_r (A^r \otimes {}^rA) \to \bigoplus_r (A_r \otimes A^r), \qquad a \otimes b \mapsto (a \otimes 1)\Delta(b),$
- $\mathrm{can}_r^c : \bigoplus_r (A^r \otimes {}^rA) \to \bigoplus_r ({}^rA \otimes {}_rA), \qquad a \otimes b \mapsto \Delta^{op}(a)(1 \otimes b),$
- $\mathrm{can}_l^c : \bigoplus_r (A_r) \otimes {}_rA) \to \bigoplus_r (A^r \otimes A_r), \qquad a \otimes b \mapsto (a \otimes 1)\Delta^{op}(b)$

Second result

Proposition

Any regular I-partial multiplier Hopf algebra (A, Δ) is a regular weak multiplier Hopf algebra, cf. [VDW13]

Corollary (Local units)

Any regular I-partial multiplier Hopf algebra admits local units.

The Antipode S

Let (A, Δ) be a regular I-partial multiplier bialgebra.

Definition (Antipode)

A bijective I-partial algebra anti-morphism $S : A \to A$ satisfying:

$$\forall r, s \in I, a \in A_s, b \in {}^rA, c \in A :$$
$$S(\mathbf{1}_s) = \mathbf{1}^s, \qquad S(\mathbf{1}^r) = \mathbf{1}_r,$$
$$S(a_{(1)})a_{(2)}c = \varepsilon(a)\mathbf{1}_s c, \qquad cb_{(1)}S(b_{(2)} = \varepsilon(b)c\mathbf{1}^r,$$

is called an antipode for (A, Δ)

Properties of S

Proposition

A regular I-partial multiplier bialgebra is a regular I-partial Hopf algebra if and only if it admits an antipode.

Remark

- An antipode is uniquely determined.
- The antipode satisfies the following identities:

$$S(a)_{(1)} \otimes S(a)_{(2)} = S\left(a_{(2)}\right) \otimes S\left(a_{(1)}\right),$$

$$\varepsilon \circ S = \varepsilon.$$

- The antipodes of A^{op} and A^{cop} are given by S^{-1}.

Third Result

Proposition

I-partial multiplier Hopf algebras are examples of regular multiplier Hopf algebroids, cf. [Tim16].

Invariant functionals

Let (A, Δ) be a regular I-partial multiplier bialgebra.

Definition (left/right invariant functionals)

A functional $\varphi : A \to \mathbb{C}$ is called *left invariant* if its support lies in $\bigoplus_{r,s} {}_s^r A_s^r$ and if the following invariance condition holds for all $a \in A$:

$$(\mathrm{id} \otimes \varphi)\Delta(a) = \sum_r \varphi(\mathbf{1}^r a)\, \mathbf{1}^r = \sum_r \varphi(a\mathbf{1}^r)\, \mathbf{1}^r.$$

A functional $\psi : A \to \mathbb{C}$ is called *right invariant* if its support lies in $\bigoplus_{r,s} {}_s^r A_s^r$ and if the following invariance condition holds for all $a \in A$:

$$(\mathrm{id} \otimes \psi)\Delta(a) = \sum_s \psi(\mathbf{1}_s a)\, \mathbf{1}_s = \sum_s \psi(a\mathbf{1}_s)\mathbf{1}_s.$$

Partial algebraic quantum groups

Definition (PAQG)

A regular I-partial multiplier Hopf algebra, for which there exist a left invariant functional φ such that for all $s \in I$, $\varphi(A_s) \neq 0$ and a right invariant functional ψ such that for all $s \in I$, $\psi(A^s) \neq 0$, is called an *I-partial algebraic quantum group*.

Remark

The extra condition on φ and ψ is equivalent to their faithfulness.

Fourth result

Proposition

I-partial algebraic quantum groups are examples of algebraic quantum groupoids, cf. [VDW17].
Hence, they are examples of measured regular multiplier Hopf algebroids, cf. [Tim16].

The modular automorphisms

Remark

If A has a faithful *left* invariant functional φ, then $\varphi_S := \varphi \circ S$ and $\varphi_{S^{-1}} := \varphi \circ S^{-1}$ are both *right* invariant functionals for A

Proposition

There exist modular automorphisms for any I-partial algebraic quantum group A. We denote them as $\sigma^\varphi, \sigma^\psi$.
They are algebra automorphisms of A, uniquely determined by asking that for all $a, b \in A$, they satisfy:

$$\varphi(ab) = \varphi(b\sigma^\varphi(a)), \qquad \psi(ab) = \psi(b\sigma^\psi(a))$$

The modular element

Proposition

There exists a modular element for A, denoted δ_φ. It is a multiplier of A, uniquely determined by asking that for all $a \in A$, it satisfies:

$$\varphi_S(a) = \varphi(a\delta_\varphi), \qquad \varphi_{S^{-1}}(a) = \varphi(\delta_\varphi a)$$

The (restricted) dual algebra $\check{A}$

Definition

The (restricted) dual of A, denoted $\check{A}$, is defined to be the vector space $\check{A} := \{\varphi(a-)|a \in A\}$ with the convolution product, defined by setting, for all $\omega, \chi \in \check{A}$,
$(\omega \cdot \chi)(a) = \chi((\omega \otimes \mathrm{id})\Delta(a)) = \omega((\mathrm{id} \otimes \chi)\Delta(a))$

Remark

We have that:

$$\check{A} = \{\varphi(-a)|a \in A\} = \{\psi(a-)|a \in A\} = \{\psi(-a)|a \in A\}.$$

Remark

Its multiplier algebra $M(\check{A})$ is characterised by:

$$M(\check{A}) = \{\omega \in A^*|\forall a \in A : (\omega \otimes \mathrm{id})\Delta(a), (\mathrm{id} \otimes \omega)\Delta(a) \in A\}$$

The comultiplication $\check{\Delta}$

Definition

The comultiplication on $\check{A}$, denoted by $\check{\Delta}$ is determined uniquely by setting, for all $\omega, \chi \in \check{A}$, $a, b \in A$:

$$\Big(\check{\Delta}(\omega)(\chi \otimes 1), a \otimes b\Big) = \omega(b \cdot \underbrace{(\mathrm{id} \otimes \chi)\Delta(a)}_{\in A}),$$

$$\Big((1 \otimes \chi)\check{\Delta}(\omega), a \otimes b\Big) = \omega(\underbrace{(\chi \otimes \mathrm{id})\Delta(b)}_{\in A} \cdot a).$$

Duality result

- We endow $\check{A}$ with a left and right I^2-grading as follows:

$$ {}^{r}_{t}\check{A}^{s}_{u} = \left\{ \omega \left(\mathbf{1}\begin{pmatrix} t \\ u \end{pmatrix} - \mathbf{1}\begin{pmatrix} r \\ s \end{pmatrix} \right) \middle| \, \omega \in \check{A} \right\}. $$

- $\check{A}$ has (faithful) left and right invariant functionals. For $\omega = \varphi(-a), \chi = \psi(b-) \in A$ we have:

$$ \check{\varphi}(\omega) = \epsilon(a), \qquad \check{\psi}(\chi) = \epsilon(b). $$

Proposition (Duality)

The dual of an I-partial algebraic quantum group is again an I-partial algebraic quantum group.

Biduality

Let (A, Δ) be an I-partial algebraic quantum group with fixed left and right invariant functionals φ and ψ.

Proposition

There is an isomorphism between (A, Δ) and the dual PAQG of $(\check{A}, \check{\Delta})$, given by:

$$A \xrightarrow{\cong} \check{\check{A}}, \qquad a \mapsto \check{\varphi}\left(-\varphi(-a)\right).$$

Under this isomorphism:

$$\varphi = \check{\check{\varphi}}, \qquad \psi = \check{\check{\psi}}.$$

I-partial $*$-algebraic quantum groups

Definition (I-Partial $*$-algebra)

An I-partial algebra $\bigoplus_{r,s} {}_rA_s$ such that the total algebra A is a $*$-algebra and such that $({}_rA_s)^* = {}_sA_r$, is called an *I-partial $*$-algebra*.

Definition (I-partial multiplier Hopf $*$-algebra)

An I-partial multiplier Hopf algebra (A, Δ), such that the algebra A is an I^2-partial $*$-algebra and such that the comultiplication $\Delta : A \to \widetilde{M}(A \otimes^I A) \subset M(A \otimes A)$ is a morphism of $*$-algebras, is called an *I-partial multiplier Hopf $*$-algebra*.

Properties of I-Partial multiplier Hopf $*$-Algebras

Remark (positivity)

A functional ω on an I-partial $*$-algebra is called *positive* if for all $a \in A$:

$$\omega(a * a) \geq 0.$$

It is automatically self-adjoint:

$$\omega(a^*) = \overline{\omega(a)}$$

Proposition

The antipode of an I-partial multiplier Hopf $*$-algebra satisfies, for all $a \in A$:

$$S(S(a)^*)^* = a. \tag{1}$$

I-partial $*$-algebraic quantum groups

Definition (I-P*AQGs)

An I-partial multiplier Hopf $*$-algebra, which has a faithful left invariant functional φ such that φ is positive, is called an *I-partial $*$-algebraic quantum group*.

Remark

Any I-P*AQG also has a positive, faithful, right invariant functional ψ. We can in fact take $\psi = \varphi_S$. This follows from one of our main theorems.

Proposition (Duality)

The modular element of an I-partial $*$-algebraic quantum group is self-adjoint:

$$\delta_\varphi^* = \delta_\varphi.$$

The dual I-partial $*$-algebraic quantum group $(\check{A}, \check{\Delta})$

- We may define a $*$-structure on $\check{A}$ by setting, for all $\omega \in \check{A}$ and for all $a \in A$:

$$\omega^*(a) := \overline{\omega(S(a)^*)}.$$

Proposition

$(\check{A}, \check{\Delta})$ becomes an I-partial $*$-algebraic quantum group for the $*$-structure defined above.

Corollary (Biduality)

The double dual of an I-partial $*$-algebraic quantum group is isomorphic to the original I-partial $*$-algebraic quantum group.

Main Theorem I

Let (A, Δ) be an I-partial $*$-algebraic quantum group and let φ be a fixed positive, faithful, left invariant functional for (A, Δ).

Theorem (Simultaneous diagonalisability)

The automorphisms σ^φ, S^2 and left and right multiplication by δ_φ are jointly diagonalisable, with all eigenvalues positive real numbers. There is a faithful right invariant functional:

$$\psi = \varphi_S = \varphi_{S^{-1}}.$$

Representation theory

Definition

- A module for an I-partial multiplier Hopf algebra (A, Δ) is a unital A-module (i.e. $A \cdot V = V$.)
- A unitary module for an I-partial multiplier Hopf $*$-algebra $(A, \Delta, *)$ is a module for (A, Δ) on a pre-Hilbert space V, such that for all $a \in A$ and for all $v, w \in V$ we have:

$$\langle v, aw \rangle = \langle a^* v, w \rangle .$$

- A $*$-representation of A is a Hilbert space $\mathcal{H}$, together with a $*$-homomorphism $A \to B(\mathcal{H})$, such that $A\mathcal{H}$ is dense in $\mathcal{H}$.

Remark

An I-partial $*$-algebraic quantum group (A, Δ) acts by bounded operators on its unitary modules V.

I-P*AQGs of compact/discrete type

- We say an I-partial $*$-algebraic quantum group is of *compact type* if it contains the base units $\mathbf{1}\binom{r}{s}$.
- We call its underlying I-partial multiplier Hopf algebra an *I-partial Hopf algebra*.
- We say an I-partial $*$-algebraic quantum group is of *discrete type* if $(\check{A}, \check{\Delta})$ is of compact type.

Proposition

The notion of an I-partial $*$-algebraic quantum group of compact type coincides with that of a partial compact quantum group, cf. [DCT15]

Remark

We call an I-partial $*$-algebraic quantum group of discrete type a *partial discrete quantum group*.

Normalization of φ

- If $\mathbf{1}\binom{r}{s} \in A$, then we can ask $\varphi(\mathbf{1}\binom{r}{s}) = 1$ for all $r \sim s$. We then call φ a *normalized*, positive, left invariant functional.
- If we assume φ to be normalized as defined above, it is automatically also right invariant. We call it an *invariant integral.*
- It is also normalized as faithful positive right invariant functional, and $\varphi = \varphi \circ S$. In particular also $\delta_\varphi = 1$.

Conventions for comodules

- Let $V = \underset{r,s}{\oplus} {}_rV_s$ be an I-bigraded vector space.
- We view V as a bimodule for the algebra $Fun(I) = \{f : I \to \mathbb{C}\}$, by setting, for $v \in {}_rV_s$:

$$\lambda : Fun(I) \times V \to V, \qquad (f, v) \mapsto f \cdot v = f(r)v \qquad (v \in {}_rV)$$

$$\rho : V \times Fun(I) \to V, \qquad (v, g) \to v \cdot g = g(s)v \qquad (v \in V_s)$$

- If V is a pre-Hilbert space, we will assume that the ${}_rV_s$ are mutually orthogonal.
- Denote by ${}_r\Delta_s(a)$ the component of $\Delta(a)$ which lives in ${}_rA_s \otimes {}^rA^s$: ${}_r\Delta_s(a) = (\mathbf{1}_r \otimes \mathbf{1}^r)\Delta(a)(\mathbf{1}_s \otimes \mathbf{1}^s)$

Comodules for an I-PAQG

Let (A, Δ) be an I-partial Hopf algebra.

Definition

An (A, Δ)-comodule is an I-bigraded vector space V and a collection of linear maps $\delta = \{{}_r\delta_s : V \to V \otimes A\}_{r,s \in I}$, such that:

- For all $k, l, r, s \in I$: ${}_r\delta_s\left({}_kV_l\right) \subseteq {}_rV_s \otimes {}^r_kA^s_l$,
- For all $k, l, r, s \in I$, we have:

$$(\mathrm{id} \otimes {}_k\Delta_l)\,{}_r\delta_s = ({}_r\delta_s \otimes \mathrm{id})\,{}_k\delta_l \qquad (\mathrm{id} \otimes \varepsilon)\,{}_r\delta_s = \lambda(\mathbf{1}_r)\rho(\mathbf{1}_s)$$

- For any $v \in V$ and any $s \in I$ fixed, there are only finitely many $r \in I$ such that ${}_r\delta_s(v) \neq 0$.
- For any $v \in V$ and any $r \in I$ fixed, there are only finitely many $s \in I$ such that ${}_r\delta_s(v) \neq 0$.

The associated corepresentation X

- We use the sumless Sweedler notation for ${}_r\delta_s$:

$${}_r\delta_s(v) = v_{(0;rs)} \otimes v_{(1;rs)}.$$

- Because of the final two conditions, the following maps are well-defined. (Their evaluation on any fixed $v \in V$ is a finite sum.)

$$\delta_s = \sum_r {}_r\delta_s, \qquad {}_r\delta = \sum_s {}_r\delta_s.$$

- The linear map X, defined as:

$$X : V \otimes A \to V \otimes A, \qquad v \otimes a \mapsto \sum_{r,s} {}_r\delta_s(v)(1 \otimes a)$$

is then called the associated *corepresentation*.

Unitary comodules for an I-P*AQG

Let A be an I-partial Hopf $*$-algebra. Let V be a pre-Hilbert space, with inner product $\langle -,-\rangle$. We define an A-valued inner product $\langle -,-\rangle_A$ on $V \otimes A$ by setting, for all $a, b \in A$ and for all $v, w \in V$:

$$\langle v \otimes a, w \otimes b\rangle_A = \langle v, w\rangle a^* b.$$

Definition

We say a comodule (V, δ) over an I-partial Hopf $*$-algebra (A, Δ) is *unitary* if V is a pre-Hilbert space and if the associated corepresentation X is isometric, i.e.:

$$\langle X(v \otimes a), X(w \otimes b)\rangle_A = \sum_r \langle v\mathbf{1}_r, w\mathbf{1}_r\rangle a^* \mathbf{1}_r b.$$

Equivalence of $\check{A}$-modules and A-comodules

Let (A, Δ) be a partial algebraic quantum group of compact type, let V be an I-bigraded vector space.

Proposition

There is a one-to-one correspondence between left $\check{A}$-module structures on V and right A-comodule structures δ on V. The $\check{A}$-module structure on V is determined, for $v \in V$ and $\omega \in \check{A}$ by setting:

$$\omega \cdot v = \sum_{r,s} (\mathrm{id} \otimes \omega)_r \delta_s(v).$$

If V is an I-bigraded pre-Hilbert space, then this correspondence sends unitary A-comodules to unitary $\check{A}$-modules and vice versa.

The algebra $\mathscr{U}$

Let (A, Δ) be an I-partial algebraic quantum group of compact type.

Definition

The algebra $\mathscr{U}(A)$ is defined as the space of linear functionals $\chi : A \to \mathbb{C}$ such that, for a fixed r and s, the number of indices t, u such that $\chi\left(\mathbf{1}\binom{r}{t} - \mathbf{1}\binom{s}{u}\right) \neq 0$ or $\chi\left(\mathbf{1}\binom{t}{r} - \mathbf{1}\binom{u}{s}\right) \neq 0$, is finite.

The product on $\mathscr{U}(A)$ is defined by setting, for any $a \in A$:

$$(\omega \cdot \chi)(a) = (\omega \otimes \chi)\Delta(a).$$

Remark

We have $\mathscr{U}(A) \subseteq M(\check{A}) \subseteq A^*$. This algebra contains the unit ε.

The Drinfeld pre-double

Remark

We denote $\check{\mathbf{1}}\binom{r}{s} = \varepsilon(\mathbf{1}_s - \mathbf{1}_r)$ for the base units of $\mathscr{U}(A), \check{A}$.
Then we have:

$$\check{\mathbf{1}}\binom{r}{s}\omega = \omega(\mathbf{1}^s - \mathbf{1}^r) \qquad \omega\check{\mathbf{1}}\binom{r}{s} = \omega(\mathbf{1}_s - \mathbf{1}_r)$$

Definition

We define $\mathscr{D}(A)$ as thye universal unital algebra generated by A and $\mathscr{U}(A)$, with relations $\mathbf{1}\binom{r}{s} = \check{\mathbf{1}}\binom{r}{s}$ and interchange relations

$$\omega a = a_{(2)}\omega(a_{(3)} - S^{-1}(a_{(1)})).$$

The algebra $\mathcal{D}(A)$

Definition

We define $\mathcal{D}(A) = A\check{A} \subseteq \mathscr{D}(A)$.

Remark

- It is in fact a subalgebra of $\mathscr{D}(A)$,
- The following multiplication maps are isomorphisms:

$$\bigoplus_{r,s} \left(A_s^r \otimes {}_s^r\check{A} \right) \to \mathcal{D}(A), \qquad a \otimes \omega \mapsto a\omega$$

$$\bigoplus_{r,s} \left(\check{A}_s^r \otimes {}_s^rA \right) \to \mathcal{D}(A), \qquad \omega \otimes a \mapsto \omega a$$

- The algebra $\mathcal{D}(A)$ is a non-degenerate, idempotent I^2-algebra with the grading ${}_t^r\mathcal{D}(A)_u^s = \mathbf{1}\binom{r}{t}\mathcal{D}(A)\check{\mathbf{1}}\binom{s}{u}$.

The Drinfeld double

Definition

The coproduct on the Drinfeld double $\mathcal{D}(A)$ is defined by:

$$\Delta_{\mathcal{D}} : \mathcal{D}(A) \to M(\mathcal{D}(A) \otimes^I \mathcal{D}(A)); \qquad a\omega \mapsto \Delta(a)\check{\Delta}(\omega).$$

Theorem

With the coproduct defined above, $(\mathcal{D}(A), \Delta_{\mathcal{D}})$ becomes an I-partial algebraic quantum group. The antipode is given by:

$$S_{\mathcal{D}}(a\omega) = \check{S}(\omega)S(a),$$

The invariant functionals are given, for $a \in A^r_s, {}^r_s\check{A}$

$$\varphi_{\mathcal{D}} : \mathcal{D}(A) \to \mathbb{C}, \qquad a\omega \mapsto \varphi(a)\check{\varphi}(\omega),$$

$$\psi_{\mathcal{D}} : \mathcal{D}(A) \to \mathbb{C}, \qquad a\omega \mapsto \psi(a)\check{\psi}(\omega).$$

The modular and $*$-structure on $\mathcal{D}(A)$

- The modular structure on $\mathcal{D}(A)$ for all $a \in A$ and $\omega \in \check{A}$, is given by:

$$\sigma^{\varphi_{\mathcal{D}}}(a) = S^2(a), \qquad \sigma^{\varphi_{\mathcal{D}}}(\omega) = \check{S}^2(\omega)$$
$$\delta_{\varphi_{\mathcal{D}}} = \check{\delta}_{\check{\varphi}}\delta_{\varphi}\nu, \qquad \nu_{\mathcal{D}} = 1$$

- If (A, Δ) was an I-partial $*$-algebraic quantum group, then the $*$-structure on $(\mathcal{D}(A), \Delta_{\mathcal{D}})$ is defined by:

$$(a\omega)^* = \omega^* a^*, \qquad (\omega a)^* = a^* \omega^*.$$

- For this $*$-structure, the Drinfeld double becomes an I-partial $*$-algebraic quantum group.

Yetter-Drinfeld modules for I-PAQGs

Definition

An I-bigraded vector space V with a left A-module structure and a right A-comodule structure δ is called a *Yetter-Drinfeld module* if the following compatibility holds for all $a \in A$ and $v \in V$:

$$ {}_r\delta_s(av) = \sum_{pq} \mathbf{1}\begin{pmatrix} s \\ r \end{pmatrix} a_{(2)} v_{(0;pq)} \otimes \mathbf{1}^r a_{(3)} v_{(1;pq)} S^{-1}(a_{(1)}) \mathbf{1}^s $$

We call the Yetter-Drinfeld module *unitary* if V is an I-bigraded pre-Hilbert space and if the module and comodule structure are unitary.

Characterisation of Yetter-Drinfeld modules

Let V be an I-bigraded vector space. Let (A, Δ) be an I-partial algebraic quantum group of compact type.

Theorem

There is a one-to-one correspondence between Yetter-Drinfeld A-module structures on V and left $\mathcal{D}(A)$-module structures on V by setting, for all $a \in A, \omega \in \check{A}$ and $v \in V$:

$$(a\omega) \cdot v = a(\mathrm{id} \otimes \omega)\delta(v).$$

If (A, Δ) is a partial compact quantum group (i.e. if it has a compatible $*$-structure) then this correspondence induces a one-to-one correspondence between unitary Yetter-Drinfeld modules and unitary $\mathcal{D}(A)$-modules.

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